Contents

1 Introduction **7**
1.1 Early science and mathematics 7
1.1.1 The Pythagorean theorem 9
1.1.2 What is science? 10
1.1.3 What is mathematics? 11
1.1.4 Early physics as mathematics: Back to Pythagoras 13
1.2 Modern mathematics 14
1.2.1 Science meets mathematics 14
1.2.2 Three Streams from the Pythagorean theorem 21

2 Stream 1: Number Systems **25**
2.1 The taxonomy of numbers: $\mathbb{P}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{F}, \mathbb{I}, \mathbb{R}, \mathbb{C}$ 26
2.1.1 Numerical taxonomy 31
2.1.2 Applications of integers 32
2.2 Problems NS-1 37
2.3 The role of physics in mathematics 41
2.3.1 The three streams of mathematics 42
2.3.2 Stream 1: Prime number theorems 42
2.3.3 Stream 2: Fundamental theorem of algebra 44
2.3.4 Stream 3: Fundamental theorems of calculus 45
2.3.5 Other key mathematical theorems 45
2.4 Applications of prime numbers 46
2.4.1 The importance of prime numbers 46
2.4.2 Two fundamental theorems of primes 48
2.4.3 Greatest common divisor (Euclidean algorithm) 49
2.4.4 Continued fraction algorithm 53
2.5 Problems NS-2 58
2.5.1 Pythagorean triplets (Euclid's formula) 64
2.5.2 Pell's equation 65
2.5.3 Fibonacci sequence 67
2.5.4 Diagonalization of a matrix (eigenvalue/eigenvector decomposition) . . 69
2.5.5 Finding the eigenvalues: 70
2.5.6 Finding the eigenvectors: 70
2.6 Problems NS-3 71

3 Algebraic Equations: Stream 2 **77**
3.1 Algebra and geometry as physics 77
3.1.1 The first algebra 81

3.1.2 Finding roots of polynomials . . . 81
3.1.3 Newton's method for finding the roots of $P_N(s)$: . . . 84
3.1.4 Matrix formulation of the polynomial . . . 91
3.1.5 Working with polynomials in Matlab/Octave . . . 92
3.2 Eigenanalysis . . . 94
3.2.1 Eigenvalues of a matrix . . . 94
3.2.2 Cauchy's theorem and eigenmodes . . . 96
3.2.3 Taylor series . . . 98
3.2.4 Complex–Analytic functions $\in; C$ . . . 104
3.2.5 Brune impedances . . . 106
3.2.6 Complex–analytic functions . . . 106
3.3 Problems AE-1 . . . 109
3.3.1 Analytic functions . . . 110
3.3.2 Convolution . . . 113
3.4 Root classification . . . 116
3.4.1 Convolution of monomials . . . 116
3.4.2 Residue expansions of rational functions . . . 119
3.5 Introduction to analytic geometry . . . 121
3.5.1 Merging the concepts . . . 121
3.5.2 Generalized wedge product . . . 126
3.5.3 Physical examples . . . 126
3.5.4 Development of Analytic Geometry . . . 127
3.5.5 Applications of scalar products . . . 129
3.5.6 Gaussian Elimination . . . 131
3.6 Matrix algebra of systems . . . 133
3.6.1 Vectors . . . 133
3.6.2 Vector products . . . 133
3.6.3 Norms of vectors . . . 134
3.6.4 Matrices . . . 136
3.6.5 $N \times M$ complex matrices . . . 137
3.6.6 Inverse of the 2×2 matrix . . . 139
3.7 Problems AE-2 . . . 140
3.7.1 Vector algebra in $\mathbb{R}^3$ . . . 143
3.8 Transmission (ABCD) matrix composition method . . . 149
3.8.1 Thévenin parameters of a source . . . 150
3.8.2 The impedance matrix . . . 151
3.8.3 Network power relationships . . . 152
3.8.4 Ohm's law and impedance . . . 153
3.9 Signals: Fourier transforms . . . 156
3.9.1 Properties of the Fourier transform . . . 157
3.10 Systems: Laplace transforms . . . 160
3.10.1 Properties of the Laplace transform . . . 162
3.10.2 System postulates . . . 167
3.10.3 Probability . . . 169
3.11 Complex–analytic mappings (domain-coloring) . . . 171
3.11.1 The Riemann sphere . . . 174
3.11.2 Bilinear transformation . . . 175
3.12 Problems AE-3 . . . 177

3.13 System Classification 180

4 Stream 3A: Scalar Calculus 183
4.1 The beginning of modern mathematics 184
4.2 Fundamental Theorems of Scalar Calculus 185
4.2.1 The fundamental theorem of real calculus 186
4.2.2 The fundamental theorem of complex calculus (Sec. 4.2.1) 186
4.2.3 Cauchy-Riemann conditions 187
4.3 Problems DE-1 190
4.3.1 Topics of this homework: 190
4.3.2 Complex Power Series 190
4.3.3 Cauchy-Riemann Equations 191
4.3.4 Branch cuts and Riemann sheets 192
4.3.5 A Cauer synthesis of any Brune impedance 193
4.4 Complex–analytic Brune admittance 194
4.4.1 Generalized admittance/impedance 195
4.4.2 Complex–analytic impedance 196
4.4.3 Multivalued functions 201
4.5 Three Cauchy integral theorems 207
4.5.1 Cauchy's theorems for integration in the complex plane 207
4.5.2 Cauchy Integral Formula and Residue Theorem 208
4.6 Problems DE-2 210
4.6.1 Topics of this homework: 210
4.6.2 Cauchy's theorems CT-1, CT-2, CT-3 212
4.6.3 Laplace transform applications 215
4.6.4 Computer exercises with Matlab/Octave 215
4.6.5 Inverse of Riemann $\zeta(s)$ function 217
4.6.6 Quadratic forms 219
4.7 The $\mathcal{LT}$ and its inverse $\mathcal{LT}^{-1}$ 220
4.7.1 Case for negative time ($t < 0$) and causality: 221
4.7.2 Case for zero time ($t = 0$): 221
4.7.3 Case for positive time ($t > 0$) 221
4.7.4 Properties of the $\mathcal{LT}$ 222
4.7.5 Solving differential equations: 224
4.8 Problems DE-3 226
4.8.1 Topics of this homework: Brune impedance 226
4.8.2 Brune Impedance 226
4.8.3 Transmission-line analysis 227

5 Stream 3B: Vector Calculus 229
5.1 Properties of fields and potentials 229
5.1.1 Scalar and vector fields: notation 229
5.1.2 Gradient ∇, divergence $\nabla\cdot$, curl $\nabla\times$ and Laplacian ∇^2 231
5.1.3 Scalar Laplacian operator in N dimensions 235
5.2 Partial differential equations and field evolution 242
5.2.1 The Laplacian ∇^2: 242
5.2.2 Scalar wave equation (Acoustics) 245
5.2.3 The Webster horn equation (WHEN) 246

5.2.4 Matrix formulation of the WHEN . . . 248
5.3 Problems VC-1 . . . 250
5.3.1 Topics of this homework: . . . 250
5.3.2 Scalar fields and the ∇ operator . . . 250
5.3.3 Vector fields and the ∇ operator . . . 251
5.3.4 Vector algebra . . . 251
5.3.5 Vector and scalar field identities . . . 252
5.3.6 Integral theorems . . . 252
5.4 Examples of finite-length horns . . . 253
5.4.1 Uniform Horn #5 . . . 254
5.4.2 Conical horn #2 . . . 256
5.4.3 Exponential horn #3 . . . 256
5.5 Solution methods . . . 257
5.5.1 Eigenfunctions $\varrho^{\pm}(r,t)$ of the WHEN . . . 259
5.6 Integral definitions of $\nabla()$, $\nabla\cdot()$, $\nabla\times()$, and $\nabla\wedge()$ . . . 260
5.6.1 Maxwell's equations . . . 261
5.6.2 Electric fields $\boldsymbol{e},\boldsymbol{d}$ . . . 261
5.6.3 Gradient: $\boldsymbol{e}=-\nabla\phi(\boldsymbol{x})$ . . . 261
5.6.4 Divergence: $\nabla\cdot\boldsymbol{d}=\rho$ [C/m^3] . . . 262
5.6.5 Divergence and Gauss's law . . . 262
5.6.6 Magnetic fields $\boldsymbol{h},\boldsymbol{b}$ . . . 264
5.6.7 Integral definition of the curl: $\nabla\times\boldsymbol{h}=\boldsymbol{c}$ . . . 264
5.6.8 Helmholtz's decomposition theorem . . . 265
5.6.9 Second-order operators . . . 268
5.7 The unification of electricity and magnetism . . . 270
5.7.1 Maxwell's equations . . . 271
5.7.2 Derivation of the vector wave equation . . . 273
5.8 Potential solutions of Maxwell's equations . . . 275
5.9 Problems VC-2 . . . 278
5.9.1 Topics of this homework: . . . 278
5.9.2 Partial differential equations (PDEs): Wave equation . . . 278
5.9.3 Helmholtz's formula . . . 279
5.9.4 Maxwell's Equations . . . 280
5.9.5 Speed of light . . . 281
5.9.6 Application of Maxwell's equations . . . 281
5.9.7 Second-order differentials . . . 282
5.9.8 Capacitor analysis . . . 282
5.9.9 Webster horn equation . . . 283
5.10 Further readings . . . 283

Appendices **285**

A Notation **287**
A.1 Number systems . . . 287
A.1.1 Units . . . 287
A.1.2 Notation . . . 287
A.1.3 Common mathematical symbols . . . 288
A.1.4 Classification of numbers: . . . 290