Small Acoustic Tubes Revisited

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Daniel M. Warren
Senior Engineer/Scientist
Knowles Electronics, Inc.

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Introduction

Lately there has been some interest, both within the company and from customers, to make our PSpice models available in SI (or MKS) units. For most elements, this is a straightforward unit conversion. Not so, however, with the acoustic tube and magnetic elements. In this paper I will address the former.

1 The ZCK approximation

Traditionally at Knowles, we have used equations given by Zuercher, *et al* [1], hereafter referred to as *the ZCK approximation,* to calculate transmission line parameters for tubes. The ZCK approximation is a set of non-transcendental functions chosen to fit a more exact solution to the Kirchhoff equation involving Bessel functions [2, 3]. At the time, Bessel functions were expensive to calculate, and the ZCK approximation offered a factor of 40 faster calculation of transmission line parameters.

The ZCK approximations were derived in CGS units, and have worked fine as long as all acoustic modeling was in CGS. However, I encountered some difficulty in converting the equations to another unit system. As given in [1], the units are inconsistent. For example, if the functions $M_1, \ldots, M_4$ are dimensionless as stated in the text, then the real and imaginary parts of Equation 8 do not have the same units. I am sure this oversight is due to the 9 year lag between the oral presentation of the material in 1977 and the writing of the journal article in 1986.

The unit inconsistency prompted me to obtain some of the references and track down the error. In the process, I found the original Bessel function solution. By today's standards, they aren't too bad. Since I would be recalculating the tables for PSpice anyway, I decided to go ahead and use Bessel function expressions instead of the ZCK approximation.

2 The Bessel function solution

The solution is summarized here without justification. See [2, 3]. The series impedance Z and shunt admittance Y per unit length of tube are

$$Z = \frac{j\omega R_0}{c}\left[1 - F(r_v)\right]^{-1}, \tag{2.1}$$

$$Y = \frac{j\omega}{cR_0}\left[1 + (\gamma - 1)F(r_t)\right], \tag{2.2}$$

where

$$R_0 = \frac{\rho c}{\pi a^2}$$

is the characteristic impedance of the tube in the absence of viscous and thermal effects. The quantity ω is the radial frequency, c is the speed of sound in free space, ρ is the density of air, γ is the ratio of specific heats, and a is the radius of the tube. Thermodynamic properties are summarized in Table 2.1.

The function F is given in the Kirchhoff solution and is

$$F(r) = \frac{2}{\sqrt{-j}\, r}\, \frac{J_1\left(\sqrt{-j}\, r\right)}{J_0\left(\sqrt{-j}\, r\right)}.$$

The Kirchhoff function is evaluated at the nondimensional parameters

$$r_v = a\sqrt{\frac{\rho\omega}{\eta}}, \quad r_t = \nu r_v,$$

where ν is the square root of the Prandtl number

$$\nu = \sqrt{\frac{\eta C_p}{\kappa}},$$

η is the coefficient of shear viscosity, C_p is the specific heat of air at constant pressure, and κ is the coefficient of thermal conductivity.

It is now a simple matter to calculate transmission line parameters in different unit systems. The only dimensional parameters in the theory are ρ, η, and c. Their values in CGS and SI units are summarized in Table 2.2.

$\rho = 1.1769 \times 10^{-3}(1 - .00335\Delta T)\,\mathrm{g/cm^3}$
$\eta = 1.846 \times 10^{-4}(1 + .0025\Delta T)\,\mathrm{g/s\ cm}$
$\gamma = 1.4017(1 - 0.00002\Delta T)$
$\nu = 0.8410(1 - 0.0002\Delta T)$
$c = 3.4723 \times 10^4(1 + 0.00166\Delta T)\,\mathrm{cm/s}$

Table 2.1: Theromdynamic constants, where $\Delta T = T - 26.85°\mathrm{C}$. Equations are good in the range $\Delta T = \pm 10°\mathrm{C}$.

	CGS	SI
ρ	$1.120 \times 10^{-3}\,\mathrm{g/cm^3}$	$1.120\,\mathrm{kg/m^3}$
η	$1.819 \times 10^{-4}\,\mathrm{g/s\ cm}$	$1.819 \times 10^{-5}\,\mathrm{kg/s\ m}$
γ	1.402	
ν	0.8420	
c	$3.439 \times 10^4\,\mathrm{cm/s}$	$3.439 \times 10^2\,\mathrm{m/s}$

Table 2.2: Thermodynamic constants evaluated at 21 °C, given in CGS and SI units.

3 Transmission Lines

A transmission line can be defined in terms of a characteristic impedance Z_c of an infinite length of tube and its propagation wavenumber Γ such that

$$Z_c = \sqrt{Z/Y}, \tag{3.1}$$
$$\Gamma = \sqrt{ZY}. \tag{3.2}$$

This definition is useful for defining a continuous transmission line. The transfer matrix $\mathbb{T}$ and admittance matrix $\mathbb{Y}$ for a length ℓ of transmission line are

$$\mathbb{T} = \begin{bmatrix} \cosh \ell\Gamma & Z_c \sinh \ell\Gamma \\ Z_c^{-1} \sinh \ell\Gamma & \cosh \ell\Gamma \end{bmatrix}$$

and

$$\mathbb{Y} = \frac{1}{Z_c \sinh \ell\Gamma} \begin{bmatrix} \cosh \ell\Gamma & 1 \\ 1 & \cosh \ell\Gamma \end{bmatrix}$$

Alternately, the series impedance and shunt admittance are defined in terms of the series resistance R, series inductance L, shunt conductance G, and shunt capacitance C (per unit length) as

$$Z = R + j\omega L, \tag{3.3}$$
$$Y = G + j\omega C. \tag{3.4}$$

this definition is useful for specifying the transmission line in terms of short discrete segments, as in Figure 3.1. This is useful in PSpice, where we cannot specify frequency dependent L or C in `TLINE` models and are forced to build up a ladder network to represent the tube.

To construct frequency dependent elements in PSpice, use current generators with the same frequency dependence as the admittance of the element modeled. The frequency dependence is specified as `FREQ` tables in (dB, Phase) format:

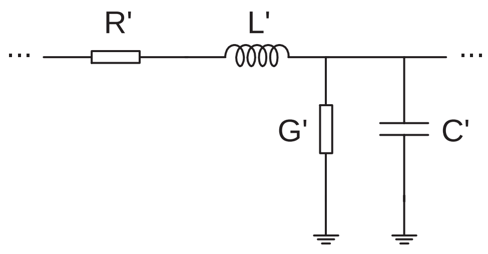

Figure 3.1: Section of a discrete model of a transmission line. Multiple sections are cascaded to simulate a long transmission line. Primes indicate values are scaled to length of segment.

$$\texttt{GR1} = \left(20 \log_{10} \frac{1}{R}, 0\right) \tag{3.5}$$
$$\texttt{GL1} = \left(20 \log_{10} \frac{1}{2\pi f L}, -90°\right) \tag{3.6}$$
$$\texttt{GR2} = (20 \log_{10} G, 0) \tag{3.7}$$
$$\texttt{GC1} = (20 \log_{10} 2\pi f C, 90°) \tag{3.8}$$

I have written programs to create PSpice `.SUBCKT` models for common tube diameters[1]. Libraries in CGS and SI units have been created for common tube diameters and are available from myself or Janice. Janice has already verified results are identical to ZCK derived models. The newer models have been extended in frequency range to 20 Hz–20 kHz.

References

[1] Joseph C. Zuercher, Elmer V. Carlson, and Mead C. Killion. Small acoustic tubes: New approximations including isothermal and viscous effects. *J. Acoust. Soc. Am.*, 83(4):1653–1660, April 1988.

[2] A. H. Benade. On the propagation of sound waves in a cylindrical conduit. *J. Acoust. Soc. Am.*, 44(2):616–623, 1968.

[3] Douglas H. Keefe. Acoustical wave propagation in cylindrical ducts: Transmission line parameter approximations for isothermal and nonisothermal boundary condtions. *J. Acoust. Soc. Am.*, 75(1):58–62, January 1984.

[1] These programs are not in a format suitable for wide distribution, unless you have *Mathematica* and Perl.